Sec 1.4 Separable ODEs.

ODEs $\frac{dy}{dx} = f(x,y)$ in which f can be "split"

as $f(x,y) = g(x) \cdot h(y)$.

Ex: Find a gen sol. for $\frac{dy}{dx} = -6xy$

$$\frac{dy}{dx} = (-6x)(y) \Rightarrow \int \frac{dy}{y} \overset{*}{=} \int -6x\,dx$$

(* assume $y \neq 0$ for now)

$$\Rightarrow \ln|y| = -3x^2 + C$$

$$\Rightarrow |y| = e^{-3x^2 + C}$$

$$\Rightarrow y = \underbrace{\boxed{\pm e^C}}_{\text{"A"}} e^{-3x^2}$$

since $y \neq 0$, for now assume $A \neq 0$.

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* is $y(x) \equiv 0$ a sol. ($\equiv$ means "equal everywhere") to the ODE? Yes, because $y \equiv 0 \Rightarrow y' = 0$,

so $\frac{dy}{dx} \overset{?}{=} (-6x)(y)$ $\quad 0 = -6x \cdot 0 = 0$ ✓

$\Rightarrow$ $y = Ae^{-3x^2}$ is a gen. sol, where A can be any real num. Also, because it is "$y(x) = \cdots$", it is an explicit sol.

Side note: Equations $\frac{dy}{dx} = \frac{12}{x^2+16}$ and $y' = y(y-3)$ are separable...

- $\frac{dy}{dx} = \underbrace{\left(\frac{12}{x^2+16}\right)}_{g(x)} \underbrace{(1)}_{h(y)} \longrightarrow \int \frac{dy}{1} = \int \frac{12}{x^2+16}\,dx$

  $y + C_1 = \cdots$

- $\frac{dy}{dx} = \underbrace{(1)}_{g} \underbrace{(y(y-3))}_{h} \Rightarrow \frac{dy}{y(y-3)} = 1\,dx \cdots$

"Natural Growth / Decay Eq: $\frac{dx}{dt} = kx$ ★Read Ex 3 & 4 in Book

Ex: Newton's Law of Cooling $\frac{dT}{dt} = k(A-T)$ $(k>0)$

$A = 400$, $k = 3/2$, $T(0) = 60$

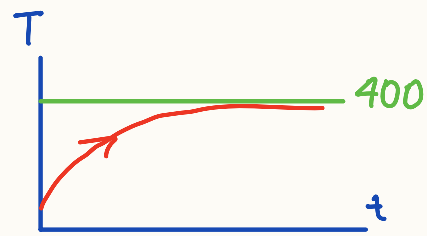
$$-\ln|400-T| = \frac{3}{2}t + c$$

$$\ln|400-T| = -\frac{3}{2}t + C$$

$$|400-T| = e^{-(3/2)t + c}$$

$$400 - T = \pm e^{c} e^{-3t/2}$$

$$T = 400 - Ce^{-3t/2}$$ Gen sol.



Need $60 = T(0) = 400 - Ce^{0}$
$= 400 - C$

$\Rightarrow C = 340$, and $T(t) = 400 - 340e^{-3t/2}$

is the part. sol. (it is again explicit, $T(t) = \cdots$)

Ex: Find an "implicit" general solution for

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5}.$$

Form $F(x,y) = C$

(note we cannot have $y = \pm\sqrt{5/3}$ $\Rightarrow$ solutions cannot pass line $y = \pm\sqrt{5/3}$)

$$\int (3y^2-5)\,dy = \int (4-2x)\,dx$$

$$y^3 - 5y + c_1 = 4x - x^2 + c_2$$

$$y^3 - 5y + x^2 - 4x = C$$

Because this has form $F(x,y) = C$, (and not $y = \cdots$)
it is a (general) implicit solution

(not always possible to make explicit sol.)

What is a part. <u>implicit</u> sol. such that $y(1)=3$?
(pass through $(1,3)$)

$$3^3 - 5(3) + 1 - 4 = C \Rightarrow C = 9.$$

So, part. (implicit) sol. is $\underline{y^3 - 5y + x^2 - 4x = 9}$

<u>Ex</u>: $\{x^2+y^2=1,$ with $y>0\}$ anc
$\{y = +\sqrt{1-x^2}\}$ both describe a semicircle

<u>VOCAB</u>

	(general) unknown const	(particular) no unknown const
form $y=f(x)$ (explicit)	<u>Gen expl.</u> • $y = Ae^{-3x^2}$ • $y = \tan(x^3 + C)$	<u>Part. expl.</u> • $y = 2e^{-3x^2}$ • $y = \tan(x^3 + \pi/4)$
form $F(x,y)=C$ (implicit)	<u>Gen. impl.</u> • $y^3 - 5y + x^2 - 4x = C$ • $y^2 + x^2 = r^2$	<u>Part. impl.</u> • $y^3 - 5y + x^2 - 4x = 9$ • $x^2 + y^2 = 4$

Ex: **Newton's Law of Cooling:**

Find temp func. $T(t)$ with (Amb.) $A = 400°$, $k = \frac{3}{2}$,
and $T(0) = 60°$.

Recall ODE for this: $\frac{dT}{dt} = k(A - T)$

So $\left\{ \frac{dT}{dt} = \frac{3}{2}(400 - T), \; T(0) = 60 \right\}$, want part. sol.

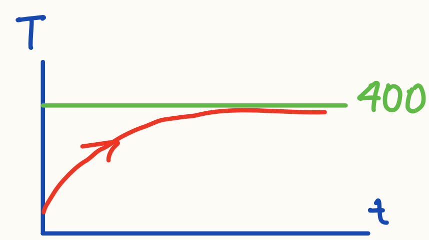
Separable, $\int \frac{dT}{(400-T)} = \int \frac{3}{2}\, dt$

$$-\ln|400 - T| = \frac{3}{2}t + c$$

$$\ln|400 - T| = -\frac{3}{2}t + C$$

$$|400 - T| = e^{-(3/2)t + c}$$

$$400 - T = \underbrace{\pm e^{c}}_{\text{"A"}}\, e^{-3t/2}$$



$\underline{T = 400 - Ae^{-3t/2}}$ Gen sol.

Need $60 = T(0) = 400 - Ae^{0}$
$= 400 - A$

$\Longrightarrow A = 340$, and $T(t) = 400 - 340e^{-3t/2}$

is the part. sol. (it is again explicit, $T(t) = \cdots$)